J. Fluid Mech. (1986), *vol.* 173, *pp.* 545–556
Printed in Great Britain

Asymptotic theory of convection in a rotating, cylindrical annulus

By F. H. BUSSE

University of Bayreuth, D-8580 Bayreuth, West Germany

(Received 25 March 1986)

The problem of convection in a rotating cylindrical annulus heated from the outside and cooled from the inside is considered in the limit of high rotation rates. The constraint of rotation enforces the two-dimensional character of the motion when the angle of inclination of the axisymmetric end surfaces with respect to the equatorial plane is small. Even when the angle of inclination is large only the dependences on the radial and the azimuthal coordinates need to be considered. The dependence on time at the onset of convection is similar to that of Rossby waves. But at higher Rayleigh numbers a transition to vacillating solutions occurs. In the limit of high rotation rates simple equations can be derived which permit the reproduction and extension of previous numerical results.

1. Introduction

Thermal convection in rotating spherical fluid shells is one of the basic physical problems of interest to astrophysicists and planetary scientists. Although the linear theory for this problem has been reasonably well understood – for reviews we refer to Eltayeb (1981) and Busse (1982) – the nonlinear properties of the problem have barely been studied. Fortunately, much can be learned about the problem in spherical shells from the simpler problems of convection in a layer with a vertical axis of rotation and in a layer with an axis of rotation at a right angle to gravity. The latter problem can be realized in the form of a rotating cylindrical annulus in which the centrifugal force replaces gravity. The annulus model has thus become the primary tool for the investigation of convection in the equatorial region of spherical shells. In fact, the equatorial region encompasses the entire region outside the cylindrical surface touching the inner spherical boundary at its equator.

The problem of convection in a layer with a vertical axis of rotation describes convection in the polar regions of spherical shells. This problem is of lesser importance since convection in the polar region is more strongly impeded by the Coriolis force than in the equatorial region. The problem is also less urgent because a number of studies of the nonlinear properties of convection in layers with a vertical axis of rotation have already been done (see Busse 1982 for references).

A basic feature of convection in a rotating cylindrical annulus bounded by inclined surfaces in the axial direction is the wave-like propagation of the columnar motions in the azimuthal direction. This time dependence gives rise to phase shifts between the radial velocity component and the perturbation of the temperature field which in turn lead to new types of instabilities which do not occur in the case of parallel end surfaces or in non-rotating layers. These instabilities tend to break the remaining symmetries of the problem and introduce new features such as an asymmetric mean

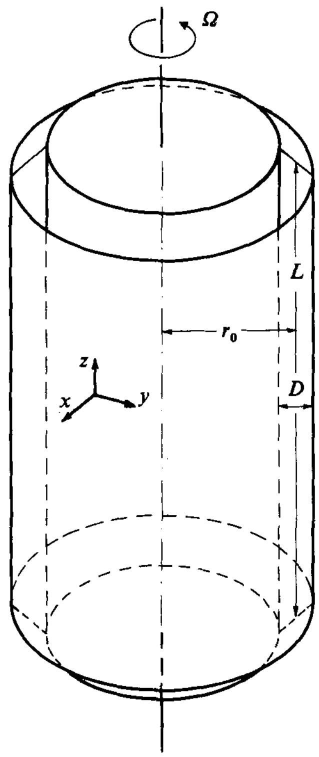

FIGURE 1. Geometrical configuration of the rotating cylindrical annulus.

zonal flow or a vacillating time dependence which appears to give rise to a chaotic time dependence at higher Rayleigh numbers. These phenomena have been found in the numerical analysis of the nonlinear problem by Or & Busse (1986, hereinafter referred to as OB). In the present paper an asymptotic theory capable of describing these features is outlined.

2. Mathematical formulation of the problem

We consider a fluid-filled cylindrical annulus rotating about its axis of symmetry with angular velocity Ω as shown in figure 1. The inner and outer cylindrical walls are kept at the constant temperatures $T_0 \pm \frac{1}{2}\Delta T$ respectively, such that a density gradient opposite to the direction of the centrifugal force is established as the basic state of the system. In application to planetary problems, the analogous buoyancy effect would be provided by the opposite density gradient since the component of gravity perpendicular to the axis of rotation is in the opposite direction to the centrifugal force. For our laboratory application, the gravity force acting parallel to the vertical axis of rotation could be taken into account (Busse 1970), but it has little effect as long as the centrifugal force is of the same order as or larger than gravity.

Using the gap width D of the annulus as the lengthscale, D^2/ν as the timescale, where ν is the kinematic viscosity, and $P\Delta T$ as the temperature scale, where P is the Prandtl number, the Navier–Stokes equations of motion and the heat equation for the deviation θ from the basic state of pure conduction can be written in the form

$$\frac{\partial}{\partial t}\boldsymbol{v}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}+2E^{-1}\boldsymbol{k}\times\boldsymbol{v} = -\nabla\pi - R\boldsymbol{i}\theta+\nabla^2\boldsymbol{v}, \tag{2.1a}$$

$$\nabla\cdot\boldsymbol{v} = 0, \tag{2.1b}$$

$$P\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)\theta = -\boldsymbol{i}\cdot\boldsymbol{v}+\nabla^2\theta. \tag{2.1c}$$

The unit vectors $\boldsymbol{i}$ and $\boldsymbol{k}$ point in radial and axial directions. The Ekman, Prandtl and Rayleigh numbers are defined by

$$E \equiv \frac{\nu}{D^2\Omega}, \quad P\equiv\frac{\nu}{\kappa}, \quad R\equiv\frac{\gamma D^3\Omega^2 r_0\Delta T}{\nu\kappa} \tag{2.2}$$

respectively, where κ is the thermal diffusivity, γ is the positive coefficient of thermal expansion, and r_0 is the mean radius. The Boussinesq approximation has been used in that the variation of density is only taken into account in connection with the body force term. In Appendix A a formulation of the problem with strong variations of density will be given. In order to study the problem in its most simple mathematical form, we assume the small-gap approximation, $D/r_0 \ll 1$. Thus we have already neglected spatial variations of the centrifugal force and of the temperature gradient of the static state in writing (2.1). We also introduce a Cartesian system of coordinates with the x- and z-coordinates in the directions of $\boldsymbol{i}$ and $\boldsymbol{k}$ and the y-coordinate in the azimuthal direction.

In order to eliminate (2.1b) we introduce the following general representation for a solenoidal velocity field:

$$\boldsymbol{v} = \nabla\times(\nabla\times\boldsymbol{k}\phi)+\nabla\times\boldsymbol{k}\psi. \tag{2.3}$$

By operating with $\boldsymbol{k}\cdot\nabla\times$ and $\boldsymbol{k}\cdot\nabla\times(\nabla\times\ \ldots)$ on (2.1a) we obtain two scalar equations for the functions ϕ, ψ,

$$\frac{\partial}{\partial t}\Delta_2\psi-\boldsymbol{k}\cdot\nabla\times[\boldsymbol{v}\cdot\nabla\boldsymbol{v}]-2E^{-1}\partial_z\Delta_2\phi = -R\partial_y\theta+\nabla^2\Delta_2\psi, \tag{2.4a}$$

$$\frac{\partial}{\partial t}\nabla^2\Delta_2\phi+\boldsymbol{k}\cdot\nabla\times(\nabla\times[\boldsymbol{v}\cdot\nabla\boldsymbol{v}])+2E^{-1}\partial_z\Delta_2\psi = -R\partial^2_{xz}\theta+\nabla^2\Delta_2\phi, \tag{2.4b}$$

where ∂_y indicates the derivative with respect to y and Δ_2 denotes $\nabla^2-(\boldsymbol{k}\cdot\nabla)^2$. The heat equation (2.1$c$) can be rewritten in the form

$$P\left[\frac{\partial}{\partial t}\theta+\nabla\times(\nabla\times\boldsymbol{k}\phi)\cdot\nabla\theta+\nabla\theta\times\nabla\psi\cdot\boldsymbol{k}\right] = -\partial_y\psi-\partial^2_{xz}\phi+\nabla^2\theta. \tag{2.4c}$$

The boundary condition that the normal velocity component and the temperature θ vanish on the cylindrical walls can be written in the form

$$\partial_y\psi+\partial^2_{xz}\phi = \theta = 0 \quad \text{at } x = \pm\tfrac{1}{2}. \tag{2.5a}$$

At the conical surfaces bounding the annular domain in the axial direction insulating boundaries will be assumed,

$$\eta_0(\partial_y\psi+\partial^2_{xz}\phi)\mp\Delta_2\phi = \eta_0\partial_x\theta\pm\partial_z\theta = 0 \quad \text{at } z = \pm\frac{L}{2D}, \tag{2.5b}$$

where η_0 is the tangent of the angle χ_B between the conical surfaces and the equatorial plane of symmetry. The axial length L of the annular region is large compared with D in typical applications, in which case the particular boundary condition (2.5*b*) for θ becomes unimportant. A finitely conducting boundary, for example, will change the solution θ only throughout a boundary layer of thickness D at most. In the following we shall discuss the linearized version of the problem (2.4), (2.5) first in the case of small η_0; the extension to the case of finite η_0 is given in appendix B.

3. The case of slightly inclined end surfaces

According to the Taylor–Proudman theorem nearly steady motions of small amplitude relative to a rotating system must be nearly independent of the z-coordinate in the direction of the rotation axis unless viscous friction is comparable with the Coriolis force. This property suggests the ansatz

$$\psi = \psi_0(x, y, t) + \check{\psi}, \tag{3.1}$$

where $\check{\psi}$ as well as ϕ are of the order η_0 which is assumed to be small in this section. Neglecting nonlinear terms we find from (2.4*a, c*) after averaging over z and using the boundary condition (2.5*b*)

$$\left[\left(\frac{\partial}{\partial t} - \Delta_2\right)\Delta_2 - \eta^* \partial_y\right]\psi_0 + R_0 \partial_y \theta_0 = 0. \tag{3.2a}$$

$$\left(P\frac{\partial}{\partial t} - \Delta_2\right)\theta_0 + \partial_y \psi_0 = 0, \tag{3.2b}$$

where η^* is defined by

$$\eta^* = \frac{4\eta_0 D}{LE}. \tag{3.3}$$

Equations (3.2) are solved by

$$\psi_0 = \sin \pi(x + \tfrac{1}{2}) \exp\{\mathrm{i}\alpha y + (\mathrm{i}\omega + \sigma) t\}, \quad \theta_0 = \psi_0 \frac{-\mathrm{i}\alpha}{P(\mathrm{i}\omega + \sigma) + \pi^2 + \alpha^2} \tag{3.4}$$

provided the dispersion relation

$$(P(\mathrm{i}\omega + \sigma) + \alpha^2 + \pi^2)\,[(\mathrm{i}\omega + \sigma + \pi^2 + \alpha^2)(\pi^2 + \alpha^2) + \mathrm{i}\alpha\eta^*] - R_0 \alpha^2 = 0 \tag{3.5}$$

is satisfied. It is of interest to inspect this equation in the case when dissipative effects are neglected, i.e. when $|\omega| \gg \alpha^2 + \pi^2$. The complex growth rate $\sigma + \mathrm{i}\omega$ obeys the equation

$$\sigma + \mathrm{i}\omega = \mathrm{i}\omega_1 \pm (R_0 \alpha^2 P^{-1} (\pi^2 + \alpha^2)^{-1} - \omega_1^2)^{\frac{1}{2}} \quad \text{with}\, \omega_1 \equiv \frac{-\alpha\eta^*}{2(\pi^2 + \alpha^2)}, \tag{3.6a, b}$$

which indicates that growing solutions only exist if R_0 exceeds the value

$$R_1 \equiv \omega_1^2 P(\pi^2 + \alpha^2)\,\alpha^{-2}. \tag{3.7}$$

For $R_0 < R_1$ two types of stable waves exist, the thermal mode and the hydrodynamic mode. The latter mode corresponds to the familiar Rossby wave with $\omega = 2\omega_1$ in the limit $R_0 = 0$, while the frequency of the thermal mode vanishes at $R_0 = 0$. In that limit the buoyancy of a steady field θ balances the ageostrophic part of the Coriolis force. As the density stratification changes from the 'unstable' (i.e. $R_0 > 0$) regime to the 'stable' regime ($R_0 < 0$) the phase relationship between θ and the radial velocity component changes by 180°. In the present context we are primarily interested in the growing solution for $R_0 > R_1$ which we call an inertial buoyancy wave. This wave resembles the Rossby wave, but differs from it in that half of the restoring force provided by the vortex-stretching term is balanced by thermal buoyancy. Thus the frequency ω_1 is only half the frequency of Rossby waves.

The neglect of the dissipative terms is not justified in general since they may be important even in the limit of vanishing diffusivities ν and κ. Double-diffusive effects, which are important in the case of density stratifications owing to the opposite effects of salt and temperature gradients (Turner 1973), also play a role in the case of convection in a rotating annulus. From the general solution of (3.5) in the marginal case $\sigma = 0$,

$$\omega_0 = \frac{-\alpha\eta^*}{(1+P)(\pi^2+\alpha^2)}, \quad R_0 = (\pi^2+\alpha^2)^3\alpha^{-2} + \left(\frac{\eta^* P}{1+P}\right)^2 \Big/ (\pi^2+\alpha^2), \qquad (3.8a, b)$$

it is obvious that for large as well as for small Prandtl numbers R_0 is generally much lower than indicated by (3.6*b*).

Only for $P = 1$ do the factors multiplying η^{*2} in (3.6*b*) and (3.8*b*) become identical, which is a characteristic property of double-diffusive phenomena. Since in the limit $P \rightarrow 0$ the dispersion relation (3.7) for Rossby waves is recovered from (3.8*a*) the term thermal Rossby waves has been introduced for convection in a rotating annulus. In Appendix B it will be shown that (3.8) continue to hold in the case of finite inclinations η_0 of the end surface. Only the definitions of R_0 and η^* must be slightly modified.

From the physical point of view the minimum R_c of the Rayleigh number as a function of α is of primary interest. In the asymptotic case of large η^* the minimizing values α_c of the wave number and R_c are given by

$$\alpha_c = \eta_P^{\frac{1}{3}}(1 - \tfrac{7}{12}\pi^2\eta_P^{-\frac{2}{3}} + \ldots); \quad R_c = \eta_P^{\frac{4}{3}}(3 + \pi^2\eta_P^{-\frac{2}{3}} + \ldots), \qquad (3.9)$$

where η_P is defined by

$$\eta_P \equiv \frac{\eta^* P}{\surd 2(1+P)}.$$

It is of interest to note that these relationships for α_c, R_c become independent of π in the limit $\eta^* = \infty$ (Busse 1970). Different modes corresponding to different radial dependences,

$$\psi^{(n)} \propto \sin n\pi (x + \tfrac{1}{2}), \quad n = 1, 2, \ldots, \qquad (3.10)$$

thus give rise to the same relationships (3.9) for $\eta^* \simeq \infty$. We shall return to this point in §4. All these modes satisfy stress-free boundary conditions at $x = \pm\frac{1}{2}$. But only minor changes will occur if rigid boundaries are assumed instead, as has been discussed in earlier work (Busse 1970; Busse & Or 1986*a*).

4. Nonlinear properties

The asymptotic relationships (3.9) indicate two remarkable features of convection in a rapidly rotating annulus. First, the azimuthal wavenumber diverges like $\eta^{*\frac{1}{3}}$ as η^* tends to infinity. Since the inhibiting influence of the Coriolis force is proportional to α it could have been expected that modes with a long wavelength in the azimuthal direction are preferred. But the buoyancy force increases with α^2 and thus a motion is preferred whose streamlines deviate strongly from the geostrophic contours or lines of constant height of the system. The only strictly geostrophic modes of motion have the form of a differential rotation, of course, and the question arises whether such a mode for which the inhibiting influence of the Coriolis force vanishes can be induced by thermal convection.

Before answering this question in the affirmative we discuss a second property of relationships (3.9), namely the vanishing dependence of the onset of convection on the gap width D. Since R_c becomes proportional to α_c^4 the temperature gradient $\Delta T/D$ for the onset of convection becomes independent of D. This property is the basic reason for the fact that the annulus model provides a good description for convection in spheres and in spherical shells where the sidewall boundaries are generally far removed. Connected with this property is the already mentioned fact that the relationships (3.9) converge in the limit $\eta^* \to \infty$ for modes with radial dependences (3.10). This feature suggests that interactions between modes of different radial symmetry may become important and that additional bifurcations may occur as the Rayleigh number is increased slightly above the critical value. In considering the nonlinear extensions of (3.2) in the case of slightly inclined boundaries

$$\left[\frac{\partial}{\partial t}+\partial_y \psi\, \partial_x - \partial_x \psi\, \partial_y\right] \Delta_2 \psi - \eta^* \partial_y \psi - \Delta_2^2 \psi + R\, \partial_y \theta = 0, \tag{4.1a}$$

$$P\left[\frac{\partial}{\partial t}+\partial_y \psi\, \partial_x - \partial_x \psi\, \partial_y\right] \theta + \partial_y \psi - \Delta_2 \theta = 0, \tag{4.1b}$$

we may distinguish two classes among the solutions for which ψ is periodic and antisymmetric in the variable $y^* \equiv y - ct$, with suitably determined drift rate c. The two types of solutions can be characterized by their symmetry properties:

$$\text{type I} \quad \psi(x, y^*) = \psi\left(-x, \frac{\pi}{\alpha} - y^*\right), \tag{4.2a}$$

$$\text{type II} \quad \psi(x, y^*) = -\psi(-x, y^*), \tag{4.2b}$$

where $2\pi/\alpha$ is the periodicity interval of the solutions. The distinction (4.2) remains valid if no-slip boundary conditions are imposed at $x = \pm\frac{1}{2}$. Among the linear modes (3.10) those with odd n correspond to type I while those with even n give rise to solutions of type II. In order to study symmetry-breaking bifurcations we start with the ansatz

$$\psi_0 = A \sin \alpha(y-ct) \sin \pi(x+\tfrac{1}{2}) + B \sin \beta(y - \hat{c}t + \chi) \sin 2\pi(x+\tfrac{1}{2}), \tag{4.3}$$

where the amplitudes A, B are assumed to be small and when χ is an arbitrary angle describing the phase shift between the two components of ψ_0. As long as the wavenumber α and β differ, χ can be neglected since it vanishes after an appropriate shift of the y-coordinate. But in the special case $\alpha = \beta$, χ becomes an important parameter of the problem.

Expression (4.3) is the first term in an expansion

$$\psi = \psi_0 + \psi_1 + \psi_2 + \dots, \tag{4.4}$$

where ψ_1 represents terms proportional to $A^\nu B^{2-\nu}$, $\nu = 0, 1, 2$, ψ_2 represents cubic terms and so on. An analogous expansion is assumed for θ with θ_0 given by the solution of the linear problem corresponding to (4.1),

$$\theta_0 = A\alpha \sin \pi(x+\tfrac{1}{2}) \frac{P\alpha c \sin \alpha(y-ct) - (\pi^2+\alpha^2) \cos \alpha(y-ct)}{P^2 c^2 \alpha^2 + (\pi^2+\alpha^2)^2}$$
$$+ B\beta \sin 2\pi(x+\tfrac{1}{2}) \frac{P\beta\hat{c} \sin[\beta(y-\hat{c}t)+\chi] - (\pi^2+\beta^2) \cos[\beta(y-\hat{c}t)+\chi]}{P^2 \hat{c}^2 \beta^2 + (\pi^2+\beta^2)^2} \tag{4.5}$$

We introduce the ansatz (4.4) into the nonlinear equations (4.1) and solve the inhomogeneous linear problem for ψ_1, θ_1. The solvability conditions in the cubic order of (4.1) provide the following relationships for R, A, B, c and $\hat{c}$:

$$[R-R_0-f_0A^2-(g_0+f_1(P))B^2]A = 0, \tag{4.6a}$$

$$[R-\hat{R}_0-g_0B^2-(f_0+g_1(P))A^2]B = 0, \tag{4.6b}$$

$$(c+\omega_0\alpha-h_1B^2)A = 0, \quad (\hat{c}+\hat{\omega}_0\beta-k_1A^2)B = 0, \tag{4.6c,d}$$

where $\hat{R}_0$ and $\hat{\omega}_0$ are given by (3.8) for R_0 and ω_0 respectively, after α has been replaced by β and π has been replaced by 2π. In writing (4.6*c*, *d*) we have used the property established in Busse & Or (1986*a*) that the term proportional to A^2 in (4.6*c*) and the term proportional to B^2 in (4.6*d*) vanish. The constants f_0, g_0 are given by

$$f_0 = \frac{P^2\alpha^2(\pi^2+\alpha^2)}{8[\omega_0^2P^2+(\pi^2+\alpha^2)^2]}, \quad g_0 = \frac{P^2\beta^2(4\pi^2+\beta^2)}{8[\hat{\omega}_0^2P^2+(4\pi^2+\beta^2)^2]}. \tag{4.7}$$

The expressions for f_1, g_1, h_1, k_1 are rather lengthy and will not be given here.

In the special case $\alpha = \beta$ more interesting relationships are obtained in place of (4.6). Time-independent relationships can only be obtained if the conditions $c = \hat{c}$ holds, since otherwise terms proportional to $\cos[2\alpha(\hat{c}-c)t-2\chi]$ or $\sin[2\alpha(\hat{c}-c)t-2\chi]$ will appear in the solvability equations. This additional condition can be met since the angle χ enters the problem as an explicit parameter,

$$[R-R_0-f_0A^2-(g_0+f_1(P)+f_2(P)\cos 2\chi+f_3(P)\sin 2\chi)B^2]A = 0, \tag{4.8a}$$

$$[R-\hat{R}_0-g_0B^2-(f_0+g_1(P)+g_2(P)\cos 2\chi+g_3(P)\sin 2\chi)A^2]B = 0, \tag{4.8b}$$

$$c+\omega_0\alpha = (h_1+h_2\cos 2\chi+h_3\sin 2\chi)B^2, \quad c+\hat{\omega}_0\alpha = (k_1+k_2\cos 2\chi+k_3\sin 2\chi)A^2. \tag{4.8c,d}$$

Besides the 'pure' solutions

$$A^2 = (R-R_0)f_0^{-1}, \quad c = -\omega_0\alpha, \quad B = 0, \tag{4.9a}$$

$$B^2 = (R-\hat{R}_0)g_0^{-1}, \quad \hat{c} = -\hat{\omega}_0\beta, \quad A = 0, \tag{4.9b}$$

there are 'mixed' solutions

$$A^2 = \frac{(f_1+f_2\cos 2\chi+f_3\sin 2\chi)(R-R_0)+g_0(R_0-\hat{R}_0)}{N} \tag{4.10a}$$

$$B^2 = \frac{(g_1+g_2\cos 2\chi+g_3\sin 2\chi)(R-\hat{R}_0)+f_0(\hat{R}-R_0)}{N} \tag{4.10b}$$

where N is given by

$$N \equiv (g_0+f_1+f_2\cos 2\chi+f_3\sin 2\chi)(f_0+g_1+g_2\cos 2\chi+g_3\sin 2\chi)-g_0f_0. \tag{4.11}$$

Solutions (4.9) and (4.10) solve (4.8*a*, *b*) for $\alpha = \beta$. In the case of the mixed solution (4.10) the drift rate c and the angle χ are determined by (4.8*c*, *d*). Solutions (4.9) and (4.10) also solve (4.6) for $\alpha \neq \beta$ if $\cos 2\chi$ and $\sin 2\chi$ are replaced by zero.

The mathematical structure of problems (4.6) and (4.8) is similar to the codimension-2-bifurcations treated by Segel (1962), by Knobloch & Guckenheimer (1983) and by Busse & Or (1986*b*) in the case of ordinary Rayleigh–Bénard convection with stress-free boundaries. A detailed discussion of the bifurcation structure of the solutions outlined above and their stability as a function of the parameters of the problem will be given in a forthcoming paper by Busse & Lin. Here we focus attention on the interpretation of the numerical results of OB. They indicate that the bifurcation of the mixed solution (4.10) from the pure solution (4.9*a*) occurs at some Rayleigh number greater than R_0. At the same point the mixed solution becomes stable and solution (4.9a) becomes unstable. Mixed solutions with $\alpha \neq \beta$ have not been found as stable solutions in the study of OB. Instead the stability analysis of OB indicates that solutions involving a third term in the representation (4.3) of the form $\check{B} \sin[\check{\beta}(y-\check{c}t)+\chi]$ with $\check{\beta} = 2\alpha-\beta$ become stable in place of the pure solution (4.8a) in some parts of the parameter space, especially for large Prandtl numbers P.

The mixed solution in the case $\alpha = \beta$ has been called the 'mean flow solution' in OB because it is characterized by a strong mean zonal flow. This component of motion is described by the y-independent component $\bar{\psi}_1$ of ψ_1, which is proportional to AB and is symmetric in x and thus describes a differential rotation which is antisymmetric with respect to the midplane of the layer. There are actually two physically different mixed solutions with opposite signs of AB since expressions (4.10) do not specify the signs of A or B and since only one of these signs can be changed by an appropriate translation in the y-direction. Since these and other properties of the mixed solutions such as the heat transport have already been discussed in OB there is no need to repeat this discussion. At the values of η^* for which the numerical computations have been carried out, the agreement with the perturbation theory outlined in this section is to within a few percent.

The transition from the pure solution to the vacillating solution occurs in competition to the transition to the meanflow solution. The numerical results of OB indicate that the vacillating solution can be described asymptotically by (4.8) if A, B, χ and c are assumed to be time dependent and if the terms with time derivatives obtained in the solvability conditions are added to the equations. Also of particular interest are vacillating solutions with $\alpha \neq \beta$ which appear to be slightly preferred according to OB. These solutions can be accommodated in the perturbation approach if the term $\check{B}(t) \sin(\beta(y-ct)+\chi(t))$ is added to (4.3) as suggested above.

In introducing the ansatz (4.3) we did not include explicitly modes of the form (3.10) with $n \geq 3$. Those modes are included implicitly, however, in the expansion scheme (4.4) since they enter the solutions through the contributions of the order A^n for odd $n \geq 3$ and B^n for even $n \geq 4$. The fact that these modes have asymptotic critical Rayleigh numbers only slightly above those for $n = 1, 2$ is likely to reduce the range of convergence of the series (4.4) as η^* becomes very large. Eventually (4.3) should be replaced by an infinite sum of all modes. But the close agreement with the numerical results of OB indicates that (4.3) provides a good approximation for solutions of (4.1) for a wide range of the parameter space.

5. Conclusion

Convection in a rotating cylindrical annulus offers a particularly simple system for the study of the transition to complex time dependence in fluid flow. Because the spatial dependence is essentially two-dimensional and periodic conditions can be assumed in the azimuthal direction, the analysis of the transitions to more complex dynamics does not pose the difficulties encountered in other turbulent fluid systems. The numerical analysis of OB has demonstrated a number of new features introduced by the transitions, and the semi-analytical approach outlined in this paper promises to elucidate the mathematical properties of the bifurcating solutions, at least in the asymptotic regime of high rotation rates. Experimental studies have produced data resembling some of the theoretical results (Azouni, Bolton & Busse 1986) such as the decrease of the Nusselt number with increasing Rayleigh number found for the mean-flow solution. But improvements in the experimental techniques are needed before quantitative comparisons can be made with the theoretical predictions about the higher transitions. The opportunity for the study of a system of similar simplicity to Rayleigh–Bénard convection or Taylor vortices but with quite different properties makes the continuing investigation of thermal Rossby waves and their modifications by bifurcations desirable.

Besides the basic physical features that can be studied in the rotating-annulus problem, the possibility of comparisons with observed dynamical features on the major planets provides a second motivation. The nonlinear analyses of Busse & Or (1986*a*) and of OB have exhibited features which have only partly been anticipated in the simpler analytical models of planetary convection of Busse (1983*a*, *b*). But modifications of the basic ideas of those models have not become necessary. The essential identity of the effects of inclined boundaries and of a density gradient, which is evident from the derivation of the vorticity equation in Appendix A, has been emphasized before. Thus the non-Boussinesq effects of the planetary atmosphere will merely influence the quantitative comparison between theory and observations. On the other hand, the more recent findings suggest new interpretations for observed phenomena such as the brown barges and the row of plumes on Jupiter as has been pointed out in OB. It is expected that further progress in the study of the asymptotic model outlined in §4 will add to these interpretations.

Appendix A: Derivation of the buoyancy-driven vorticity equation in the case of significant variations in density

Since convection layers in rotating planets and stars are characterized by strong variations in density, it is of interest to consider deviations from the Boussinesq approximation. Following earlier studies (e.g. Gough 1969) we assume an anelastic approximation in which the time derivation in the equation of continuity is neglected,

$$\nabla \cdot \rho_0 \boldsymbol{u} = 0. \tag{A 1}$$

Assuming a static basic state with an arbitrary axisymmetric gravity vector field $\boldsymbol{g} = -\nabla \Phi$ we find, after operating with $\boldsymbol{k} \cdot \nabla \times$ on the Navier–Stokes equation of motion,

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right) \zeta + (2\Omega + \zeta)(\nabla \cdot \boldsymbol{u} - \boldsymbol{k} \cdot \nabla \boldsymbol{k} \cdot \boldsymbol{u}) = \boldsymbol{k} \cdot \nabla p \times \nabla \rho^{-1} + \boldsymbol{k} \cdot \nabla \times \boldsymbol{F}, \tag{A 2}$$

where the angular velocity of the rotating system is given by $\boldsymbol{k}\Omega$, $\boldsymbol{F}$ is force of viscous friction, and ζ denotes $\boldsymbol{k} \cdot \nabla \times \boldsymbol{u}$. The term $(\nabla \times \boldsymbol{u} - \boldsymbol{k}\zeta) \cdot \nabla \boldsymbol{k} \cdot \boldsymbol{u}$ has been neglected in anticipation of the dominant role of the $\boldsymbol{k}$-component of vorticity. The assumption of a basic static state

$$\frac{1}{\rho_0} \nabla p_0 + \boldsymbol{g} = 0 \tag{A 3}$$

is not well justified in most applications, since a thermal wind must be expected in planetary and stellar interiors (Busse 1981). But this additional complication will not cause any significant change in the formulation of the problem.

Using (A 3) the definitions

$$p = p_0 + \pi, \quad \rho = \rho_0(1 - \gamma_T \hat{\theta} + \gamma_p \pi), \tag{A 4}$$

where $\hat{\theta}$ and π describe the deviations of the temperature and pressure distributions from the static state we find

$$\frac{1}{\rho_0^2} \nabla \rho \times \nabla p \cdot \boldsymbol{k} = \frac{1}{\rho_0^2} \nabla \rho_0 \times \nabla \pi \boldsymbol{k} + \boldsymbol{k} \times \boldsymbol{g} \cdot \nabla (\gamma_p \pi - \gamma_T \hat{\theta}). \tag{A 5}$$

Terms that are quadratic in $\hat{\theta}, \pi$ can be neglected since the perturbations of the pressure and temperature distributions induced by convection are assumed to be small. As in the Boussinesq case (Busse 1970) we further assume that the geostrophic balance of the convection motion is approximately satisfied,

$$(\zeta+2\Omega)\,\boldsymbol{k}\times\boldsymbol{u} \approx \frac{-1}{\rho_0}\nabla\pi, \tag{A 6}$$

where ζ is negligible in comparison with 2Ω in most cases, but has been included in order to make the argument more general. Using (A 1), (A 5) and (A 6), (A 2) can now be rewritten in the form

$$\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla\right)\zeta-(2\Omega+\zeta)\left\{\frac{1}{\rho_0}\boldsymbol{k}\cdot\nabla(\rho_0\boldsymbol{u}\cdot\boldsymbol{k})+\gamma_p\rho_0\boldsymbol{u}\cdot\boldsymbol{k}\times(\boldsymbol{g}\times\boldsymbol{k})\right\} = \gamma_T\boldsymbol{g}\times\boldsymbol{k}\cdot\nabla\hat{\theta}+\boldsymbol{k}\cdot\nabla\times\boldsymbol{F}, \tag{A 7}$$

where constant coefficients γ_T, γ_p have been assumed. Only the component of $\boldsymbol{g}$ perpendicular to the axis of rotation enters the problem and it is evident that the second term in the wavy bracket gives a contribution of the same form as the first term after the average over $\boldsymbol{k}$-direction has been taken and the boundary condition

$$\eta_0\boldsymbol{u}\cdot\boldsymbol{i}+\boldsymbol{u}\cdot\boldsymbol{k} = 0 \quad \text{at } z = \pm\frac{L}{2D}$$

has been applied. In particular the case of inward-pointing gravity gives rise to the same sign as a positive η_0 and vice versa. This property agrees with the physical picture of inward-moving fluid columns acquiring positive vorticity because of the double effects of transverse compression and of longitudinal stretching. The opposite effect occurs in outward-moving columns.

After the average over the z-coordinate in the $\boldsymbol{k}$-direction has been taken, (A 7) agrees basically with (2.4a), except for the neglect of ζ in comparison with 2Ω, and for the fact that the relationship

$$\zeta = \boldsymbol{k}\cdot\nabla\times(\rho_0^{-1}\nabla\times\boldsymbol{k}\psi) \tag{A 8}$$

replaces the simpler relationship $\zeta = -\Delta_2\psi$ of the Boussinesq case where the representation $\rho_0^{-1}\nabla\times\boldsymbol{k}\psi$ has been used for the geostrophic component of the velocity field. The equation (2.4c) hardly needs to be changed if θ is interpreted as the deviation from the distribution of potential temperature. Thus the parameter ΔT refers to the superadiabatic part of the temperature difference across the layer. In the limit of isentropic motions of an inviscid fluid, (A 7) agrees with the potential-vorticity equation derived by Glatzmaier & Gilman (1981).

Appendix B: The case of finite inclination

When η_0 assumes values of the order unity the assumption of a quasi-geostrophic velocity field (3.1) can no longer be justified. In this section we shall demonstrate, however, that the basic dynamics of convection remain unchanged. In particular a dispersion relation of the form (3.5) can still be obtained.

Assuming a y- and t-dependence of the form $\exp\{\mathrm{i}\alpha y+\mathrm{i}\omega t\}$ and assuming that ψ and ϕ satisfy the separation ansatz

$$\Delta_2\psi = -a^2\psi, \quad \Delta_2\phi = -a^2\phi, \tag{B 1}$$

we obtain from the linearized version of (2.4) the following equations:

$$-\omega a V + 2E^{-1}\partial_z W - R_0 \mathrm{i}\alpha\theta + \mathrm{i}a^3 V = 0, \tag{B 2a}$$

$$\mathrm{i}a^2\omega W + 2\mathrm{i}aE^{-1}\partial_z V + a^4 W = 0, \tag{B 2b}$$

$$\mathrm{i}\omega P\theta + a^2\theta + \frac{\alpha V}{a} = 0, \tag{B 2c}$$

where

$$V \equiv \mathrm{i}a\psi, \quad W \equiv a^2\phi \tag{B 3}$$

have been introduced as new variables and where the assumption $D \ll L$ has been made, which is well justified in typical experimental applications and which is also appropriate for applications to spherical shells (Soward 1977). The latter assumption permits the neglect of z-derivatives in comparison with x- and y-derivatives and also implies that $\partial^2_{xz}\phi$ can be neglected to first approximation in comparison with $\partial_y \psi$.

By introducing a second complex representation with the imaginary unit j and by defining

$$X \equiv V + \mathrm{j}W \tag{B 4}$$

we can combine (4.12) into a single equation

$$[\omega - \mathrm{i}a^2 + \xi \mathrm{i}\alpha^2 a^{-1} R_0(\mathrm{i}\omega P + a^2)^{-1}]X + 2E^{-1}a^{-1}\mathrm{j}\,\partial_x X = p(V, W), \tag{B 5}$$

where $p(V, W)$ is given by

$$p(V, W) \equiv \mathrm{i}\alpha^2 a^{-2} R(\mathrm{i}\omega P + a^2)^{-1}[\xi(V + \mathrm{j}W) - V]. \tag{B 6}$$

The parameter ξ will be determined by the condition

$$\langle X^* p(V, W)\rangle = 0, \tag{B 7}$$

where the angular brackets indicate the average over the fluid annulus and the asterisk indicates the complex-conjugate with respect to both imaginary units, i and j. Since we anticipate that $p(V, W)$ can be regarded as a perturbation for small Prandtl numbers P, condition (B 7) guarantees that p has a minimal influence on the eigenvalue ω, R of the left-hand side of (4.5). In other words, a suitably weighted average of the buoyancy force is taken into account in the zeroth order of the problem such that the remainder of the buoyancy force affects only properties of second order.

After neglecting the right-hand side of (B 5), we obtain the solution

$$X = \exp\{\mathrm{j}\gamma z\} \sin\pi(x + \tfrac{1}{2}) \exp\{\mathrm{i}\alpha y + \mathrm{i}\omega t\}, \tag{B 8}$$

where γ satisfies the relationship

$$(\mathrm{i}\omega P + a^2)[\mathrm{i}\omega + a^2 - \mathrm{i}2E^{-1}a^{-1}\gamma] - \alpha^2 a^{-2}\xi R_0 = 0. \tag{B 9}$$

Boundary conditions (2.5) require

$$-tg\gamma z = \frac{\eta_0 \alpha}{a} \quad \text{at } z = \frac{L}{2D}, \tag{B 10}$$

where $\partial^2_{xz}\phi$ has been neglected in comparison with $\partial_y\psi$ as mentioned above. Before dispersion relation (B 9) can be evaluated, ξ must be obtained from condition (B 7),

$$\xi = \frac{1}{2}\left(1 + \frac{D}{\gamma L}\sin\gamma\frac{L}{D}\right). \tag{B 11}$$

Using $a^2 = \alpha^2 + \pi^2$ and the definition

$$\hat{\eta} = 4D(LE)^{-1}\frac{a}{\alpha}\operatorname{arc}tg\left(\frac{\eta_0\alpha}{a}\right)$$

we write real and imaginary parts of (B 9) in the form

$$\omega = \frac{-\alpha\hat{\eta}}{(1+P)(\pi^2+\alpha^2)}, \quad \xi R_0 = (\pi^2+\alpha^2)^3\alpha^{-2} + \left(\frac{\hat{\eta}P}{1+P}\right)^2 \Big/ (\pi^2+\alpha^2), \quad \text{(B 12}a,b\text{)}$$

which exhibits a close similarity with (3.8). In the limit of vanishing η_0 the definitions of η^* and $\hat{\eta}$ coincide, ξ becomes equal to unity, and results (3.8) and (B 12) agree. As η_0 increases from zero, $|\omega|$ increases less rapidly and R_0 increases more rapidly in general than may be expected on the basis of (3.8).

For low Prandtl numbers P the real part of the right side of (B 5) is smaller than the imaginary part by a factor of the order P if the minimzing value α_c of α is used. This property can be easily checked when the asymptotic expressions analogous to (3.9) are used. Thus it is evident that neglection of $p(V, W)$ is justified in the limit of small P. At Prandtl numbers of the order unity and larger the neglect of $p(V, W)$ cannot be justified rigorously. However, the finite perturbation of (B 9) introduced by $p(V, W)$ will not change the result (B 12) in any significant way. For the application of the low-Prandtl-number expansion in the case of convection in a spherical shell see Soward (1977).

REFERENCES

AZOUNI, M. A., BOLTON. E. W. & BUSSE, F. H. 1986 Convection driven by centrifugal buoyancy in a rotating annulus. *Geophys. Astrophys. Fluid Dyn.* **34**, 301–307.

BUSSE, F. H. 1970 Thermal instabilities in rapidly rotating systems. *J. Fluid Mech.* **44**, 441–460.

BUSSE, F. H. 1981 Do Eddington–Sweet circulations exist? *Geophys. Astrophys. Fluid Dyn.* **17**, 215–235.

BUSSE, F. H. 1982 Thermal convection in rotating systems. In *Proc. 9th US National Congress of Applied Mechanics*, pp. 299–305. American Society of Mechanical Engineers, New York, NY.

BUSSE, F. H. 1983*a* A model of mean zonal flow in the major planets. *Geophys. Astrophys. Fluid Dyn.* **23**, 153–174.

BUSSE, F. H. 1983*b* Convection-driven zonal flows in the major planets. *PAGEOPH*, **121**, 375–390.

BUSSE, F. H. & OR, A. C. 1986*a* Convection in a rotating cylindrical annulus: thermal Rossby waves. *J. Fluid Mech.* **166**, 173–187.

BUSSE, F. H. & OR, A. C. 1986*b* Subharmonic and asymmetric convection rolls. *Z. angew. Math. Phys.* **37**, 608–623.

ELTAYEB, I. A. 1981 Propagation and stability of wave motions in rotating magnetic systems. *Phys. Earth Planet. Int.* **24**, 259–271.

GLATZMAIER, G. A. & GILMAN, P. A. 1981 Compressible convection in a rotating spherical shell. III. Analytic model for compressible vorticity waves. *Astrophys. J. Suppl.* **45**, 381–388.

GOUGH, D. O. 1969 The anelastic approximation for thermal convection. *J. Atmos. Sci.* **26**, 448–456.

KNOBLOCH, E. & GUCKENHEIMER, J. 1983 Convective transitions induced by a varying aspect ratio. *Phys. Rev.* A **27**, 408–417.

OR, A. C. & BUSSE, F. H. 1986 Convection in a rotating cylindrical annulus. Part 2. Transition to asymmetric and vacillating flow. *J. Fluid Mech.* **174**, 313–326.

SEGEL, L. A. 1962 The nonlinear interaction of two disturbances in the thermal convection problem. *J. Fluid Mech.* **14**, 97–114.

SOWARD, A. M. 1977 On the finite amplitude thermal instability in a rapidly rotating fluid sphere. *Geophys. Astrophys. Fluid Dyn.* **9**, 19–74.

TURNER, J. S. 1973 *Buoyancy Effects in Fluids*. Cambridge University Press.